

REMARKS ON L^p -LIMITING ABSORPTION PRINCIPLE OF SCHRÖDINGER OPERATORS AND APPLICATIONS TO SPECTRAL MULTIPLIER THEOREMS

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ABSTRACT. This paper comprises two parts. We first investigate a L^p type of limiting absorption principle for Schrödinger operators $H = -\Delta + V$ on \mathbb{R}^n ($n \geq 3$), i.e., we prove the ϵ -uniform $L^{\frac{2(n+1)}{n+3}}-L^{\frac{2(n+1)}{n-1}}$ estimates of the resolvent $(H - \lambda \pm i\epsilon)^{-1}$ for all $\lambda > 0$ under the assumptions that the potential V belongs to some integrable spaces and a spectral condition of H at zero is satisfied. As applications, we establish a sharp Hörmander type spectral multiplier theorem associated with Schrödinger operators H and deduce L^p bounds of the corresponding Bochner-Riesz operators. Next, we consider the fractional Schrödinger operator $H = (-\Delta)^\alpha + V$ ($0 < 2\alpha < n$) and prove a uniform Hardy-Littlewood-Sobolev inequality for $(-\Delta)^\alpha$, which generalizes the corresponding result of Kenig-Ruiz-Sogge [21].

1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate L^p -estimates for resolvents of Schrödinger operators $-\Delta + V$ and devote their applications to related spectral multiplier operators. Besides, we also study similar estimates for generalized Schrödinger operators $(-\Delta)^\alpha + V$ ($0 < 2\alpha < n$). Firstly, let us recall that in a paper of Kenig, Ruiz and Sogge [21], it was shown that for $n \geq 3$, there is a constant C_p independent of λ such that

$$\sup_{0 < \epsilon < 1} \|(-\Delta - (\lambda + i\epsilon))^{-1}\|_{L^p-L^{p'}} \leq C_p \lambda^{\frac{n}{2}(\frac{1}{p} - \frac{1}{p'})-1}, \quad \lambda > 0, \quad (1.1)$$

where $\frac{2n}{n+2} \leq p \leq \frac{2(n+1)}{n+3}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we mention another basic resolvent estimate for $H = -\Delta + V$ due to Agmon [1], known as *the limiting absorption principle*,

$$\sup_{0 < \epsilon < 1} \|(-\Delta + V - (\lambda + i\epsilon))^{-1}\|_{L^{2,\sigma}-L^{2,-\sigma}} \leq C(\lambda_0) \lambda^{-1}, \quad \lambda > \lambda_0 > 0, \quad (1.2)$$

where $|V(x)| \leq C(1 + |x|)^{-1-}$ and $\sigma > \frac{1}{2}$ ($L^{2,\sigma}$ is the usual weighted L^2 space). Motivated by (1.1) and (1.2), Goldberg and Schlag [13] showed a L^p version of the limiting

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absorption principle for the three dimensional Schrödinger operators. More specifically, for any given $\lambda_0 > 0$, they proved

$$\sup_{0 < \epsilon < 1} \|(-\Delta + V - (\lambda + i\epsilon))^{-1}\|_{L^{\frac{4}{3}}(\mathbb{R}^3) \rightarrow L^4(\mathbb{R}^3)} \leq C(\lambda_0)\lambda^{-\frac{1}{4}}, \quad \lambda > \lambda_0, \quad (1.3)$$

where $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$, $p > \frac{3}{2}$.

In the first part, we will address the problem that whether one can take $\lambda_0 = 0$ in (1.3). This is natural when comparing (1.3) to the free case (1.1), and it's also inspired by spectral multiplier applications (see Section 2.2 below). To this end, we mention that as far as dispersive estimates for Schrödinger equations are concerned, zero is often assumed to be neither a eigenvalue nor a resonance in order to obtain sharp decay estimates (see e.g. [18, 19, 27, 12]). In our L^p case, we need to introduce a similar condition as well.

Definition 1.1. *We say that zero is regular with respect to $H = -\Delta + V$ in $L^p(\mathbb{R}^n)$ with some $p \geq 1$, if for any $u \in L^p(\mathbb{R}^n)$ which satisfies $u = -(-\Delta)^{-1}Vu$, then $u \equiv 0$.*

Equivalently, if zero is a regular point of $H = -\Delta + V$, then the equation $-\Delta u + Vu = 0$ only has the trivial solution $u = 0$ in L^p space. Under suitable size conditions on potential V , we can prove that zero is always regular if $V \geq 0$ (see section 2.1 below). On the other hand, counterexamples can be constructed when $V < 0$. In fact, let $u(x) = (1 + |x|^2)^{-\frac{n-2}{2}}$, then it's easy to check that $u \in L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)$. A direct computation shows

$$\frac{\Delta u}{u} = -\frac{n(n-2)}{(1 + |x|^2)^2} < 0.$$

If we put $V = \frac{\Delta u}{u}$, then $V \in L^p(\mathbb{R}^n)$ for all $p \geq \frac{n}{2}$ and u satisfies that $u = -(-\Delta)^{-1}Vu$.

Under this assumption for general potential V , we can state our first main result.

Theorem 1.2. *Let $n \geq 3$ and $V \in L^{\frac{n}{2}+\sigma} \cap L^{\frac{n}{2}}$, $\sigma > 0$ be real-valued. If zero is regular with respect to $H = -\Delta + V$ in $L^{\frac{2(n+1)}{n-1}}$, then*

$$\sup_{0 < \epsilon < 1} \|(-\Delta + V - (\lambda + i\epsilon))^{-1}\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n) \rightarrow L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \leq C\lambda^{-\frac{1}{n+1}}, \quad \lambda > 0. \quad (1.4)$$

Theorem 1.2 extends estimates (1.3) of Goldberg and Schlag [13] to the limiting case $\lambda_0 = 0$ and also deals with the higher dimension case. The potential class is almost critical and it relies heavily on results of absence of imbedded eigenvalues of $-\Delta + V$ due to Ionescu and Jerison [17]. We also mention that when $V(x) = c/|x|^2$, where $c \geq -\frac{(n-2)^2}{4}$, which denotes the best constant in Hardy inequality, the uniform resolvent estimates of the form $L^{p,2} \rightarrow L^{p',2}$ in Lorentz space were obtained very recently by Mizutani [25].

As mentioned above, one motivation behind proving uniform resolvent estimates (1.4) is that they are closely related to the theory of spectral multipliers. Actually, note that (1.1) implies the following sharp estimates of the spectral measure associated with the Laplace operator

$$\|dE_{-\Delta}(\lambda)\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n) \rightarrow L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \leq C\lambda^{-\frac{1}{n+1}}, \quad \lambda > 0, \quad (1.5)$$

by the Stone formula

$$dE_{-\Delta}(\lambda) = \frac{1}{2\pi i} \left((-\Delta - (\lambda + i0))^{-1} - (-\Delta - (\lambda - i0))^{-1} \right), \quad \lambda > 0.$$

For the perturbed case $H = -\Delta + V$, similar results can also be established for H via Theorem 1.2. In particular, we shall show the following result.

Theorem 1.3. *Let $n \geq 3$, $H = -\Delta + V$ and $0 \leq V \in L^{\frac{n}{2}} \cap L^{\frac{n}{2}+\sigma}$ for some $\sigma > 0$. Then*

$$\|dE_H(\lambda)\|_{L^p-L^{p'}} \leq C\lambda^{\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})-1}, \quad \lambda > 0, \quad (1.6)$$

for all $1 \leq p \leq \frac{2(n+1)}{n+3}$.

The proof will be given in Section 2, which relies on the fact that zero is regular in $L^{\frac{2(n+1)}{n-1}}$ when $V \geq 0$. The sharpness of (1.5) indicates that the range of p in estimates (1.6) of $dE_H(\lambda)$ is also sharp. Note that in Chen, et al [4, Section 7], a smaller range $1 \leq p \leq \frac{2n}{n+2}$ was obtained based on dispersive estimates for e^{itH} . On the other hand, in Sikora, Yan and the second author [31], non-negative potentials with small enough L^p norm are required in order to obtain the optimal range of p .

In section 2, we shall apply Theorem 1.3 to establish Hörmander-type spectral multiplier theorems, which devote to L^p estimates of a spectral operator $F(H)$ initially defined in L^2 through the functional calculus

$$F(H) = \int F(\lambda) dE_H(\lambda).$$

The connection with L^p bounds of Bochner-Riesz means associated with H is also discussed, see Theorem 2.1 and Corollary 2.2.

The second part of this paper is devoted to extend (1.3) to the fractional Schrödinger operators $H = (-\Delta)^\alpha + V$, where $0 < 2\alpha < n$. Thus it's natural to first prove uniform estimates for the fractional Laplacian $(-\Delta)^\alpha$. We note that besides estimates (1.1), the following uniform Sobolev estimates obtained in Kenig, Ruiz and Sogge [21] are also closely related.

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{p,q} \|(\Delta + z)u\|_{L^p(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n), \quad z \in \mathbb{C}, \quad (1.7)$$

where

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n} \text{ and } \min \left(\left| \frac{1}{p} - \frac{1}{2} \right|, \left| \frac{1}{q} - \frac{1}{2} \right| \right) > \frac{1}{2n}. \quad (1.8)$$

(1.7) was originally motivated by certain unique continuation problems for Schrödinger operators $-\Delta + V$, which turned out to be connected with many other problems as well, see e.g. [9, 10, 11] for applications of estimating the eigenvalue bounds of Schrödinger operators. Hence it's of independent interest to extend (1.7) to more general situations. Indeed, there are lots of work concerning various generalizations on manifolds, see e.g. [2, 7, 14, 16, 23, 28, 29] and the references therein. Recently Sikora, Yan and the second author [31] proved uniform Sobolev estimates for real homogeneous elliptic operators under a non-degenerate condition. In this paper, we shall prove the following theorem.

Theorem 1.4. *Let $n \geq 3$, if $\frac{n}{n+1} \leq \alpha < \frac{n}{2}$ and $1 < p < q < \infty$ are Lebesgue exponents satisfying*

$$\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{n} \text{ and } \min\left(\left|\frac{1}{p} - \frac{1}{2}\right|, \left|\frac{1}{q} - \frac{1}{2}\right|\right) > \frac{1}{2n}, \quad (1.9)$$

then there is a uniform constant $C_{p,q} < \infty$, such that for all $z \in \mathbb{C}$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_{p,q} \|((-\Delta)^\alpha - z)u\|_{L^p(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n). \quad (1.10)$$

On the other hand, if $0 < \alpha < \frac{n}{n+1}$, then no such uniform estimates exist.

We note that if $z = 0$, then (1.10) becomes the classical Hardy-Littlewood-Sobolev inequality which is true for $0 < 2\alpha < n$ and $\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{n}$, $1 < p < q < \infty$. Our proof relies heavily on the relation between $((-\Delta)^\alpha - z)^{-1}$ and the special case $(-\Delta - z)^{-1}$ (see (3.2) and (3.9) in Section 3), whose kernel can be written explicitly. Based on this observation, we are able to obtain these estimates off the dual line $\frac{1}{p} + \frac{1}{q} = 1$ by essentially using Stein's oscillatory integral theorem (see e.g. [21, Lemma 2.2]).

The rest of the paper is organized as follows. In section 2, we shall prove Theorem 1.2 and Theorem 1.3. As applications, a sharp spectral multiplier theorem and L^p bound of Bochner-Riesz means associated with H are given. Section 3 is devoted to prove Theorem 1.4 and related estimates for $H = (-\Delta)^\alpha + V$. Throughout the paper, C and C_j denote absolute positive constants whose dependence will be specified whenever necessary. The value of C may vary from line to line.

2. LIMITING ABSORPTION PRINCIPLE AND SPECTRAL MULTIPLIER ESTIMATES

2.1. L^p -limiting absorption principle. In this subsection, we will prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. The proof is a perturbative approach, which is based on the resolvent identity that for $\epsilon > 0$ and $\lambda > 0$,

$$R_V(\lambda + i\epsilon) = (I + R_0(\lambda + i\epsilon)V)^{-1}R_0(\lambda + i\epsilon), \quad (2.1)$$

where $R_V(\lambda + i\epsilon) = (H - \lambda - i\epsilon)^{-1}$, $H = -\Delta + V$ and $R_0(\lambda + i\epsilon)$ denotes the resolvent of the free Laplacian. Since our goal is to prove the ϵ -uniform $L^{\frac{2(n+1)}{n+3}} - L^{\frac{2(n+1)}{n-1}}$ estimates of $R_V(\lambda + i\epsilon)$, it then suffices to show that the inverse $(I + R_0(z)V)^{-1}$ exists as a bounded operator on $L^{\frac{2(n+1)}{n-1}}$, and its norm is uniformly bounded for $z \in \mathbb{C}_+$, where \mathbb{C}_+ denotes the closed upper half plane \mathbb{C} . Then Theorem 1.2 will follow by combining (2.1) and resolvent estimates of the free Laplacian, i.e., the estimates (1.1). Now we divide it into the following three steps.

Step 1 (Existence of $(I + R_0(z)V)^{-1}$ on $L^{\frac{2(n+1)}{n-1}}$)

It follows from Lemma 3.1 and Lemma 3.2 in [13] that $R_0(z)V$ is a compact operator on $L^{\frac{2(n+1)}{n-1}}$ for $z \in \mathbb{C}_+$ and the inverse $(I + R_0(z)V)^{-1}$ exists and is bounded on $L^{\frac{2(n+1)}{n-1}}$ for $z \in \mathbb{C}_+ \setminus \{0\}$. Hence it remains to prove the case $z = 0$. Note that zero is assumed to be

regular in $L^{\frac{2(n+1)}{n-1}}$, then the equation $(I + (-\Delta)^{-1}V)u = 0$ only has the trivial solution $u = 0$ in $L^{\frac{2(n+1)}{n-1}}$. Hence Fredholm theorem implies that $(I + R_0(z)V)^{-1}$ is also bounded on $L^{\frac{2(n+1)}{n-1}}$ for the $z = 0$.

Step 2 (The continuity of the map $z \in \mathbb{C}_+ \mapsto R_0(z)V$)

We will prove the map $z \mapsto R_0(z)V$ is continuous in uniform operator topology from the domain $z \in \mathbb{C}_+$ to the space of bounded operators on $L^{\frac{2(n+1)}{n-1}}$. For convenience, it suffices to prove instead that $T(\lambda) = R_0(\lambda^2)V$ is continuous for λ in the first quadrant, i.e., $\operatorname{Re} \lambda, \operatorname{Im} \lambda \geq 0$. Suppose that V is bounded, and supported in the ball $\{|x| \leq R\}$. First we consider the case $\lambda \neq 0$. We set $0 < |\lambda - \mu| < \min(\frac{|\lambda|}{2}, \frac{1}{2R})$ (which implies $\lambda \neq 0$). Note that the kernel of $(-\Delta - \lambda^2)^{-1}$ satisfies $R_0(\lambda^2)(x) = |x|^{2-n}F(\lambda|x|)$, where $F(z) = z^{\frac{n-2}{2}}K_{\frac{n-2}{2}}(z)$ (see e.g. [21, p. 338]) and $K_{\frac{n-2}{2}}(z)$ denotes the modified Bessel function of the second kind. Using the fact that $|F(z)|, |F'(z)| \leq C(1 + |z|)^{\frac{n-3}{2}}$, one has

$$|R_0(\lambda^2) - R_0(\mu^2)(x, y)| \leq \begin{cases} |\lambda - \mu||x - y|^{3-n}, & \text{if } |x - y| \leq |\lambda|^{-1}, \\ |\lambda - \mu||\lambda|^{\frac{n-3}{2}}|x - y|^{\frac{3-n}{2}}, & \text{if } |\lambda|^{-1} \leq |x - y| \leq |\lambda - \mu|^{-1}, \\ |\lambda|^{\frac{n-3}{2}}|x - y|^{-\frac{n-1}{2}} & \text{if } |x - y| \geq |\lambda - \mu|^{-1}. \end{cases}$$

Then a direct calculation yields that there exist constants C_1, C_2 , depending on $\|V\|_{L^\infty}$ and R , such that

$$\|(R_0(\lambda^2) - R_0(\mu^2))Vf\|_{L^{\frac{2(n+1)}{n-1}}} \leq (C_1(V, R)|\lambda - \mu| + C_2(V, R)|\lambda - \mu|^{\frac{n-1}{2(n+1)}})\|f\|_{L^{\frac{2(n+1)}{n-1}}}.$$

Next we prove the continuity at the origin. Let $|\lambda| < \frac{1}{2R}$, we have

$$|R_0(\lambda^2) - R_0(0)(x, y)| \leq \begin{cases} |\lambda||x - y|^{3-n}, & \text{if } |x - y| \leq |\lambda|^{-1}, \\ |\lambda|^{\frac{n-3}{2}}|x - y|^{-\frac{n-1}{2}}, & \text{if } |x - y| \geq |\lambda|^{-1}, \end{cases}$$

For the region $|x| > |\lambda|^{-1}$, one has

$$\begin{aligned} & \|\chi(|x| > |\lambda|^{-1})(R_0(\lambda^2) - R_0(\mu^2))Vf(x)\|_{L^{\frac{2(n+1)}{n-1}}} \\ & \leq C(V, R)\lambda^{\frac{n-3}{2} + \frac{n-1}{2(n+1)}}\|f\|_{L^{\frac{2(n+1)}{n-1}}}, \end{aligned}$$

where χ denotes the characteristic function. For the region $|x| \leq |\lambda|^{-1}$, if $n = 3$, Hölder's inequality indicates

$$\|\chi(|x| \leq |\lambda|^{-1})(R_0(\lambda^2) - R_0(\mu^2))Vf(x)\|_{L^4} \leq C(V, R)\lambda^{\frac{1}{4}}\|f\|_{L^4},$$

if $n > 3$, notice that one can choose $r < \min(\frac{n}{n-3}, \frac{2(n+1)}{n-1})$ such that $n - 2 - \frac{n}{r} > 0$ and $|x|^{3-n} \in L^r_{loc}(\mathbb{R}^n)$, then apply Young's inequality we get

$$\|\chi(|x| \leq |\lambda|^{-1})(R_0(\lambda^2) - R_0(\mu^2))Vf(x)\|_{L^{\frac{2(n+1)}{n-1}}}$$

$$\begin{aligned}
&\leq \lambda \| |x|^{3-n} \chi(|x| \leq |\lambda|^{-1}) * Vf \|_{L^{\frac{2(n+1)}{n-1}}} \\
&\leq C(V, R) \lambda^{n-2-\frac{n}{r}} \|f\|_{L^{\frac{2(n+1)}{n-1}}},
\end{aligned}$$

which implies that the map $T(\lambda)$ is also continuous at $\lambda = 0$.

In order to pass the arguments above to the general case, we note that for any $V \in L^{n/2}$, there exists a bounded function \tilde{V} with compact support such that $\|V - \tilde{V}\|_{L^{n/2}} < \epsilon$. Then by observing the uniform Sobolev estimates (1.10), we can conclude that there exists a uniform constant C such that $\sup_{\lambda \in \mathbb{C}_+} \|R_0(\lambda^2)(V - \tilde{V})\|_{L^{\frac{2(n+1)}{n-1}} - L^{\frac{2(n+1)}{n-1}}} \leq C\|V - \tilde{V}\|_{L^{n/2}} < \epsilon$. Hence the continuity can be proved for general V by using the following equality

$$(R_0(\lambda^2) - R_0(\mu^2))V = R_0(\lambda^2)(V - \tilde{V}) + (R_0(\lambda^2) - R_0(\mu^2))\tilde{V} + R_0(\mu^2)(\tilde{V} - V).$$

Step 3 (Uniform boundedness of the norm $\|(I + R_0(z)V)^{-1}\|$ for $z \in \mathbb{C}_+$)

We have established the existence of $(I + R_0(z)V)^{-1}$, and further showed that the map $z \mapsto R_0(z)V$ is continuous from the domain $z \in \mathbb{C}_+$ to the space of bounded operators on $L^{\frac{2(n+1)}{n-1}}$. Note that for any $z, z_1 \in \mathbb{C}_+, z \neq z_1$,

$$(I + R_0(z)V)^{-1} - (I + R_0(z_1)V)^{-1} = (I + R_0(z)V)^{-1}((R_0(z_1)V - R_0(z)V)(I + R_0(z_1)V)^{-1},$$

then we can obtain that $(I + R_0(z)V)^{-1}$ is a continuous function of z on \mathbb{C}_+ . In particular, the norm $\|(I + R_0(\lambda + i\epsilon)V)^{-1}\|$ is uniformly bounded for all $0 < \epsilon \leq 1$ and $0 \leq \lambda \leq r$ with any fixed $r > 0$. On the other hand, in view of Hölder's inequality, our assumption $V \in L^p$ ($p > \frac{n}{2}$) yields that $\|V\|_{L^{2(n+1)/(n-1)} - L^r} \leq C$, where $\frac{1}{r} - \frac{n-1}{2(n+1)} = \frac{1}{p} < \frac{2}{n}$. In addition, an interpolation between estimates (1.1) and (1.10) gives $\|R_0(z)\|_{L^r - L^{2(n+1)/(n-1)}} \leq C|z|^{\frac{n}{2p}-1}$. Thus there exists some large enough constant $c_0 > 0$ such that $\|R_0(z)V\|_{L^{2(n+1)/(n-1)} - L^{2(n+1)/(n-1)}} < \frac{1}{2}$ provided $z \in \mathbb{C}_+$ with $|z| \geq c_0$. So the Neumann series expansion directly shows that

$$\sup_{|z| \geq c_0} \|(I + R_0(z)V)^{-1}\|_{L^{\frac{2(n+1)}{n-1}} - L^{\frac{2(n+1)}{n-1}}} \leq 2.$$

As a conclusion, we obtain that

$$\sup_{\lambda \in \mathbb{R}, 0 < \epsilon \leq 1} \|(I + R_0(\lambda + i\epsilon)V)^{-1}\|_{L^{\frac{2(n+1)}{n-1}} - L^{\frac{2(n+1)}{n-1}}} < \infty,$$

which completes the proof of Theorem 1.2. \square

Proof of Theorem 1.3. Note that by the Riesz-Thorin interpolation theorem, it suffices to prove (1.6) for the endpoint $p = \frac{2(n+1)}{n-1}$ and $p = 1$ respectively.

Step 1 (the case $p = \frac{2(n+1)}{n-1}$) Recall that $0 \leq V \in L^{\frac{n}{2}} \cap L^{\frac{n}{2}+\sigma}$ for some $\sigma > 0$, we first show that the assumptions on V implies that 0 is regular in $L^{\frac{2(n+1)}{n-1}}$. To this end, we may assume that there exists some $f \in L^{\frac{2(n+1)}{n-1}}$ such that $f = -(-\Delta)^{-1}Vf$. Set $g = Vf$, then Hölder's inequality indicates $\Delta f = g \in L^r$ with $\frac{1}{r} = \frac{n-1}{2(n+1)} + \frac{1}{p}$ for some $p > \frac{n}{2}$. Then from $f = -(-\Delta)^{-1}g$ and the Sobolev inequality we can conclude that $f \in L^{\frac{2(n+1)}{n-1}} \cap L^q$ for

some $q > \frac{2(n+1)}{n-1}$ by

$$\frac{1}{q} = \frac{n-1}{2(n+1)} + \frac{1}{p} - \frac{2}{n} < \frac{n-1}{2(n+1)}.$$

Now by repeating this bootstrapping procedure we can obtain $f \in L^{\frac{2(n+1)}{n-1}} \cap L^q$ for all $q > \frac{2(n+1)}{n-1}$. On the other hand, note that $V \in L^{\frac{n}{2}}$, then Hölder's inequality also gives $\Delta f = -Vf \in L^s$ by

$$\frac{1}{s} = \frac{n-1}{2(n+1)} + \frac{2}{n} < \frac{n+3}{2(n+1)},$$

which implies that $s' > \frac{2(n+1)}{n-1}$. Hence we can choose $q = s'$ such that $f \in L^{\frac{2(n+1)}{n-1}} \cap L^{s'}$ and the inner product $(\Delta f, f)$ makes sense. Besides, in view of Gagliardo-Nirenberg interpolation inequality, we can conclude that $\nabla f \in L^2$ and

$$\|\nabla u\|_{L^2} \leq C\|\Delta f\|_{L^s}^{1/2}\|f\|_{L^{s'}}^{1/2} < \infty.$$

Thus it follows that

$$0 \leq (Vf, f) = (\Delta f, f) = -\|\nabla f\|^2 \leq 0,$$

which implies $f = 0$ in this case, i.e zero is regular in $L^{\frac{2(n+1)}{n-1}}$. Hence we are allowed to apply the uniform resolvent estimates (1.4) from Theorem 1.2. Thus, by Stone's formula

$$dE_H(\lambda)f = \frac{1}{2\pi i}((H - (\lambda + i0))^{-1} - (H - (\lambda - i0))^{-1})f,$$

the estimate (1.6) is valid for $p = \frac{2(n+1)}{n+3}$.

Step 2 (the case $p = 1$) Since under our assumption on the potential V , the semigroup e^{-tH} satisfies the following Gaussian estimates

$$|e^{-tH}(x, y)| \leq Ct^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \quad (2.2)$$

for some $C > 0$, see e.g. [6]. Hence we obtain that for $1 \leq p \leq q \leq \infty$,

$$\|e^{-tH}\|_{L^p-L^q} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}, \quad t > 0. \quad (2.3)$$

According to the Laplace transform formula

$$(1 + tH)^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty e^{-utH} u^{k-1} du, \quad t > 0, \quad k > 0,$$

we have

$$\begin{aligned} \|(1 + tH)^{-k}\|_{L^p-L^q} &\leq \frac{1}{\Gamma(k)} \int_0^\infty e^{-u} (ut)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} u^{k-1} du \\ &\leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}, \end{aligned} \quad (2.4)$$

provided $k > \frac{n}{2}(\frac{1}{p}-\frac{1}{q})$. Note that $dE_H(\lambda) = 2^{2k}(1 + H/\lambda)^{-2k}dE_H(\lambda)$. Then combine (2.4) with $L^{p_0} - L^{p'_0}$ estimate of $dE_H(\lambda)$ obtained already in step 1, it follows that (see e.g. [31, Lemma 3.3])

$$\|dE_H(\lambda)\|_{L^1-L^\infty} = 2^{2k}\|(1 + H/\lambda)^{-k}dE_H(\lambda)(1 + H/\lambda)^{-k}\|_{L^1-L^\infty}$$

$$\begin{aligned}
&\leq C\|(1 + H/\lambda)^{-k}\|_{L^{p'_0-L^\infty}} \|dE_H(\lambda)\|_{L^{p_0-L^{p'_0}}} \|(1 + H/\lambda)^{-k}\|_{L^1-L^{p_0}} \\
&\leq C\lambda^{\frac{n}{2}-1}, \quad \lambda > 0.
\end{aligned} \tag{2.5}$$

Therefore the desired estimate (1.6) is valid for all $1 \leq p \leq \frac{2(n+1)}{n+3}$. \square

2.2. Applications to spectral multipliers $f(H)$. We start by recalling that via a well known T^*T argument (see e.g. Sogge [32]), spectral measure estimate (1.5) is in fact equivalent to the following Stein-Tomas theorem.

$$\left(\int_{S^{n-1}} |\hat{f}|^2 d\sigma \right)^{\frac{1}{2}} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}. \tag{2.6}$$

Such restriction type of estimates are essentially required to obtain sharp results in the theory of spectral multipliers, we refer to [4, 30, 31] and literature therein.

As mentioned before, Theorem 1.3 will be used to obtain certain spectral multiplier estimates. We remark that there are a great number of works devoted to the L^p -theory of spectral multipliers $f(H)$ of non-negative self-adjoint operators. This is a related area of harmonic analysis, which has attracted a lot of attention during the last thirty years or so. The literature devoted to the subject is so broad that it is impossible to provide complete and comprehensive bibliography. Therefore we quote only a few recent articles, which are directly related to our study, e.g. see [3, 4, 8, 15, 23, 30, 31] and therein references. Actually, for Schrödinger operator $H = -\Delta + V$, once we have the spectral measure estimate (1.6), then we have the following conclusion.

Theorem 2.1. *Suppose that $n \geq 3$, $H = -\Delta + V$ satisfies the assumptions of Theorem 1.3. For any bounded Borel function F such that $\sup_{t>0} \|\eta F(t \cdot)\|_{W_2^\alpha} < \infty$ for some $\alpha > \max\{n(\frac{1}{p} - \frac{1}{2}), \frac{1}{2}\}$ and $1 \leq p \leq \frac{2(n+1)}{n+3}$. Here $\eta \in C_0^\infty(0, \infty)$ is an arbitrary non-zero auxiliary function and $\|F\|_{W_2^\alpha} = \|(1 - \frac{d^2}{dx^2})^{\frac{\alpha}{2}} F\|_{L^2(\mathbb{R})}$. Then the operator $F(H)$ is bounded on $L^r(\mathbb{R}^n)$ for all $p < r < p'$. Furthermore, if the function F is even and compactly supported, then the operator $F(H)$ is bounded for all $p \leq r \leq p'$.*

Proof. We first recall that in [31, Proposition 2.2], it was shown that for a non-negative self-adjoint operator L on $L^2(X)$, where homogeneous space (X, μ) satisfies the doubling conditions. If H satisfies (2.2) (only Davies-Gaffney estimates are required), and estimate (1.6) is valid for $1 \leq p \leq \frac{2(n+1)}{n+3}$, then for any bounded Borel function F such that $\sup_{t>0} \|\eta F(t \cdot)\|_{W_2^\alpha} < \infty$ for some $\alpha > \max\{n(\frac{1}{p} - \frac{1}{2}), \frac{1}{2}\}$ with $1 \leq p \leq \frac{2(n+1)}{n+3}$. Then the operator $F(H)$ is bounded on $L^r(\mathbb{R}^n)$ for all $p < r < p'$. Now consider $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$, we use again the fact that under the assumption $0 \leq V \in L_{\text{loc}}^1$, the semigroup e^{-tH} satisfies the Gaussian estimates (2.2). Hence the proof is complete by applying (1.6) in Theorem 1.3. \square

When $p = 1$, Theorem 2.1 corresponds to a version of classical Hörmander Fourier multiplier theorem (see e.g. Stein [35]), which measures the needed regularities of spectral function F in Sobolev space. The variable parameter p in Theorem 2.1 is interesting

and important. A remarkable applied example of spectral multipliers is Bochner-Riesz means. Let's recall that Bochner-Riesz operators of index δ for a non-negative self-adjoint operator H are defined by

$$S_R^\delta(H) = \frac{1}{\Gamma(\delta+1)} \left(1 - \frac{H}{R}\right)_+^\delta, \quad R > 0.$$

Since $(1 - \lambda)_+^\delta \in W_2^\alpha$ if and only if $\delta > \alpha - 1/2$. Applying Theorem 2.1 above (see also [4, Coro. 3.2]), then we obtain the following result.

Corollary 2.2. *Suppose H satisfies assumptions of Theorem 2.1 and $1 \leq p \leq \frac{2(n+1)}{n+3}$. Then for $\delta > n(1/p - 1/2) - 1/2$, we have*

$$\sup_{R>0} \|S_R^\delta(H)\|_{L^p-L^p} \leq C.$$

In the free case $H = -\Delta$, it's well known that such Bochner-Riesz mean result is a consequence of Stein-Tomas restriction estimates (2.6), see e.g. Stein [35, p. 390]. Also see Sogge [33] for Riesz means on compact manifold. In the present perturbed operator $H = -\Delta + V$, the spectral measure estimates (1.6) play a similar role as restriction estimates (2.6) in the proof of Corollary 2.2, as showed in [4, 30].

Finally, we mention that in scattering theory, the following intertwining identity

$$f(H)P_c = W_\pm f(-\Delta)W_\pm^*$$

is valid for a Borel functions f , where P_c denotes the projection on the continuous spectrum and W_\pm represents the corresponding wave operators. It is well known that Yajima [36] showed that W_\pm is bounded on L^p ($1 \leq p \leq \infty$), thus based on the identity above, the L^p boundedness of $f(H)P_c(H)$ is reduced to the free case $f(-\Delta)$. However, compared to our Theorem 2.1, this wave operator approach requires quite fast decay condition for V or certain regularity depending on dimension (for example, he needed $|V| \lesssim \langle x \rangle^{-5-\epsilon}$ in three dimension).

3. LIMITING ABSORPTION FOR FRACTIONAL SCHRÖDINGER OPERATORS

3.1. Proof of Theorem 1.4 (the fractional Laplacian $(-\Delta)^\alpha$). We first prove the case $0 < \alpha < \frac{n}{n+1}$. Suppose estimate (1.10) is valid for all z , in particular, take $p = \frac{2n}{n+2\alpha}$, $q = p' = \frac{2n}{n-2\alpha}$, and choose $z = 1 - i\epsilon$ with $\epsilon > 0$, then we have

$$\|((- \Delta)^\alpha - 1 + i\epsilon)^{-1} f\|_{L^{\frac{2n}{n-2\alpha}}} \leq C \|f\|_{L^{\frac{2n}{n+2\alpha}}}. \quad (3.1)$$

Observe that

$$\operatorname{Im} \frac{1}{1 - |\xi|^{2\alpha} - i\epsilon} = \frac{\epsilon}{(1 - |\xi|^{2\alpha})^2 + \epsilon^2},$$

which converges weakly to $d\sigma_{S^{n-1}}$, the surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ as $\epsilon \rightarrow 0$. Therefore it follows from the standard TT^* arguments that

$$\left(\int_{S^{n-1}} |\hat{f}|^2 d\sigma \right)^{\frac{1}{2}} \leq C \|f\|_{L^{2n/(n+2\alpha)}}, \quad f \in L^{\frac{2n}{n+2\alpha}}.$$

However, the assumption $0 < \alpha < \frac{n}{n+1}$ implies that $\frac{2n}{n+2\alpha} > \frac{2(n+1)}{n+3}$, which contradicts to the Stein-Tomas restriction theorem (see (2.6)).

Now we shall prove that estimate (1.10) is valid under the assumption (1.9), and that $\alpha \geq \frac{n}{n+1}$. By homogeneity consideration and the gap condition $n(\frac{1}{p} - \frac{1}{q}) = 2\alpha$, we can assume from now on that $|z| = 1$. Denote by K the Schwartz kernel of the resolvent $((-\Delta)^\alpha - z)^{-1}$, it's convenient to write $K = K' + K''$, where $K'(x) = K(x)$, if $|x| \leq 1$, and 0 otherwise.

Case 1: α is an integer. Suppose $\alpha = m \in \mathbb{Z}^+$, in this case the resolvent can be expressed as

$$((-\Delta)^m - z)^{-1} f = \frac{1}{mz} \sum_{k=0}^{m-1} z_k (-\Delta - z_k)^{-1} f, \quad f \in C_0^\infty(\mathbb{R}^n), \quad (3.2)$$

where $z_k = z^{\frac{1}{m}} e^{i\frac{2k\pi}{m}}$ ($k = 0, 1, \dots, m-1$) are the k -th root of z . To estimate K' , we shall need the following expression of the Green function of $(-\Delta - z_k)^{-1}$ (see e.g. [21, p. 338])

$$(-\Delta - z_k)^{-1}(x, y) = \left(\frac{-z_k}{|x-y|^2} \right)^{\frac{n-2}{4}} K_{\frac{n-2}{2}}((-z_k)^{\frac{1}{2}}|x-y|), \quad (3.3)$$

where $K_{\frac{n-2}{2}}(z)$ has the following asymptotic expansion

$$K_{\frac{n-2}{2}}(z) = \frac{\pi \csc((n-2)\pi/2)}{2} \sum_{j=0}^{m-1} \frac{(z/2)^{2j-\frac{n}{2}+1}}{\Gamma(j-\frac{n}{2}+2)j!} + o(z^{2m-\frac{n}{2}-1}), \quad z \rightarrow 0, \quad (3.4)$$

if n is odd, and

$$K_{\frac{n-2}{2}}(z) = \frac{1}{2} \sum_{j=0}^{m-1} \frac{(-1)^j (\frac{n}{2} - j - 2)!}{j!} \left(\frac{z}{2} \right)^{2j-\frac{n}{2}+1} + o(z^{2m-\frac{n}{2}-1}), \quad z \rightarrow 0, \quad (3.5)$$

if n is even, where we have used the fact that in this case $n \geq 2m + 2$. We also note that

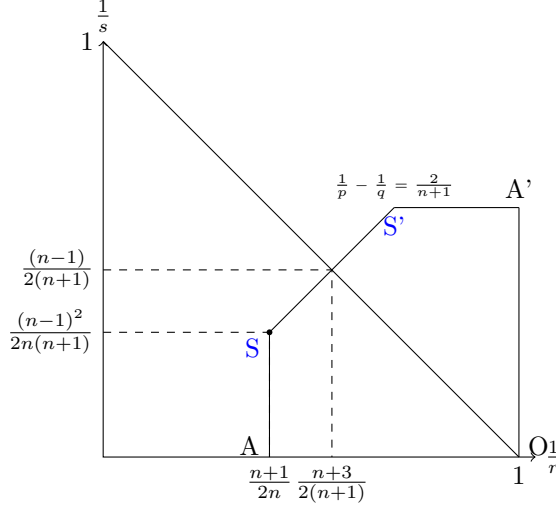
$$\sum_{k=0}^{m-1} z_k^{j+1} = \begin{cases} 0, & \text{if } j = 0, 1, \dots, m-2, \\ m, & \text{if } j = m-1. \end{cases} \quad (3.6)$$

Then it follows from (3.2)-(3.6) that

$$|K'(x)| \leq C|x|^{2m-n},$$

hence, Hardy-Littlewood-Sobolev inequality yields

$$\|K' * f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (3.7)$$

FIGURE 1. $L^p - L^q$ estimates

where $\frac{1}{p} - \frac{1}{q} = \frac{2m}{n}$. Thanks to the expression (3.2), it follows immediately from Stein's oscillatory integral theorem in [21, Lemma 2.4] that

$$\|K'' * f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (3.8)$$

where $(\frac{1}{p}, \frac{1}{q})$ lies either on the open line segment SS' or in the interior of the pentagon $SAOA'S'$ in the Figure below. Combining (3.7) and (3.8), we prove the estimates (1.10) in this case.

Case 2: α is not a integer. We shall first prove the particular situation $\frac{n}{n+1} \leq \alpha < 1$, then show that the general case can be obtained after a slight modification.

To simplify matters we point out here that we only need to prove estimates (3.3) for $z = e^{i\theta}$, $0 < |\theta| < \frac{\pi\alpha}{2}$, since there exists a constant depending on α such that

$$\|(-\Delta)^\alpha (e^{i\theta} - (-\Delta)^\alpha)^{-1}\|_{L^1-L^1} \leq C_\alpha, \quad \frac{\pi\alpha}{2} \leq |\theta| \leq \pi,$$

which follows from the fact that the semigroup $e^{-t(-\Delta)^\alpha}$ can be extended to an analytic semigroup on L^1 in the region $C = \{z : \operatorname{Re} z > 0\}$ and its operator norm is uniformly bounded in every closed subsector of C (see e.g. Komatsu [22]). Hence we are allowed to write $z = s^\alpha$, where $0 < |\arg s| < \pi$. We recall the following expression of the resolvent of the fractional power of the negative Laplacian (see Martinez, Sanz [24])

$$\begin{aligned} ((-\Delta)^\alpha - s^\alpha)^{-1} &= \frac{s^{1-\alpha}}{\alpha} (-\Delta - s)^{-1} \\ &+ \frac{\sin \alpha \pi}{\pi} \int_0^\infty \frac{\lambda^\alpha (\lambda - \Delta)^{-1}}{\lambda^{2\alpha} - 2\lambda^\alpha s^\alpha \cos \alpha \pi + s^{2\alpha}} d\lambda \\ &\triangleq R_1 + R_2. \end{aligned} \quad (3.9)$$

And we break the kernel of R_j in the same way of K such that $R_j(x) = K'_j(x) + K''_j(x)$. Hence we have $K' = K'_1 + K'_2$ and $K'' = K''_1 + K''_2$.

First, it's easy to see that K'_1 is the "good term" for estimating K' . Indeed, from (3.3), we have

$$|K'_1(x)| \leq C|x|^{2-n} \leq C|x|^{2\alpha-n}. \quad (3.10)$$

To estimate K'_2 , we recall that (see Stein [34, p. 132])

$$(\lambda - \Delta)^{-1}(x, y) = C \int_0^\infty e^{-\delta\lambda - \frac{|x-y|^2}{4\pi\delta}} \delta^{-\frac{n+2}{2}} \frac{d\delta}{\delta}, \quad \lambda > 0. \quad (3.11)$$

And a direct computation yields

$$\begin{aligned} & \int_0^\infty \frac{\lambda^\alpha (\lambda - \Delta)^{-1}(x, y)}{\lambda^{2\alpha} - 2\lambda^\alpha s^\alpha \cos \alpha\pi + s^{2\alpha}} d\lambda \\ &= \int_0^\infty e^{-\frac{|x-y|^2}{4\pi\delta}} \delta^{-\frac{n+2}{2}} \frac{d\delta}{\delta} \int_0^\infty \frac{\lambda^\alpha e^{-\delta\lambda}}{\lambda^{2\alpha} - 2\lambda^\alpha s^\alpha \cos \alpha\pi + s^{2\alpha}} d\lambda \\ &= \int_0^\infty e^{-\frac{|x-y|^2}{4\pi\delta}} \delta^{-\frac{n+2\alpha}{2}} f_\alpha(\delta) \frac{d\delta}{\delta} \\ &= C|x-y|^{2\alpha-n} + o(|x-y|^{2\alpha-n}), \quad |x-y| \rightarrow 0, \end{aligned}$$

where $f_\alpha(\delta) = \int_0^\infty \frac{t^\alpha e^{-t}}{t^{2\alpha} - 2\delta^\alpha t^\alpha s^\alpha \cos \alpha\pi + (\delta s)^{2\alpha}} dt$, and $\lim_{\delta \rightarrow 0} f_\alpha(\delta) = \Gamma(1-\alpha)$. The last equality follows by observing that

$$|x|^{-n+2\alpha} = \frac{(4\pi)^{-\frac{n+2\alpha}{2}}}{\Gamma(\frac{n-2\alpha}{2})} \int_0^\infty e^{-\frac{|x|^2}{4\pi\delta}} \delta^{-\frac{n+2\alpha}{2}} \frac{d\delta}{\delta}.$$

Thus we have

$$|K'_2(x)| \leq C|x|^{2\alpha-n}. \quad (3.12)$$

From (3.12) and (3.12), we obtain estimates (3.7) with $\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{n}$.

Next, we estimate the term K'' . Since, as indicated before, K''_1 satisfies estimates (3.8) with the same exponents there, it suffices to consider convolution with the kernel K''_2 , which we shall see is the "good term" in this case. In fact, we recall that (3.11) implies that for $\lambda > 0$

$$|(\lambda - \Delta)^{-1}(x, y)| \leq \begin{cases} C\lambda^{\frac{n-2}{4}} |x-y|^{-\frac{n-2}{2}} e^{-\sqrt{\lambda}|x-y|}, & \sqrt{\lambda}|x-y| > 1, \\ C|x-y|^{2-n}, & \sqrt{\lambda}|x-y| \leq 1. \end{cases} \quad (3.13)$$

Using this we obtain

$$\begin{aligned} & \int_0^\infty \left| \frac{\lambda^\alpha (\lambda - \Delta)^{-1}(x, y)}{\lambda^\alpha - s^\alpha e^{\pm i\pi\alpha}} \right| d\lambda \\ & \leq C|x-y|^{2-n} \int_0^{|x-y|^{-2}} \left| \frac{\lambda^\alpha}{\lambda^\alpha - s^\alpha e^{\pm i\pi\alpha}} \right| d\lambda \end{aligned}$$

$$\begin{aligned}
& + C|x-y|^{-n} \int_1^\infty \frac{t^{2\alpha+\frac{n}{2}} e^{-t}}{|t^{2\alpha} - |x-y|^{2\alpha} s^\alpha e^{\pm i\pi\alpha}|} dt \\
& \leq C|x-y|^{-n-2\alpha}, \quad \text{if } |x-y| > 1,
\end{aligned}$$

where the last inequality follows from the fact that when $|\arg s^\alpha| \leq \frac{\pi\alpha}{2}$, then

$$|\lambda^\alpha - s^\alpha e^{\pm i\pi\alpha}| \geq C_\alpha, \quad \text{if } 0 < \lambda < 1,$$

and

$$|t^{2\alpha} - |x-y|^{2\alpha} s^\alpha e^{\pm i\pi\alpha}| \geq C_\alpha |x-y|^{2\alpha}, \quad \text{if } |x-y| > 1, t > 0.$$

Hence, Young's inequality gives

$$\|K'' * f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p < q \leq \infty. \quad (3.14)$$

So K'' satisfies estimate (3.10) with the same exponents there, which prove the case $\alpha < 1$.

We are left to show the remaining case where $m < \alpha < m+1$, $m = 1, 2, \dots$, the idea is the same so we just sketch arguments. Instead of using formula (3.11), we shall use the following functional formula:

$$\begin{aligned}
((-\Delta)^\alpha - s^{\alpha_m})^{-1} f &= \frac{s^{1-\alpha_m}}{\alpha_m} ((-\Delta)^{m+1} - s)^{-1} \\
&+ \frac{\sin \alpha_m \pi}{\pi} \int_0^\infty \frac{\lambda^{\alpha_m} (\lambda + (-\Delta)^{m+1})^{-1} f}{\lambda^{2\alpha_m} - 2\lambda^{\alpha_m} s^{\alpha_m} \cos \alpha_m \pi + s^{2\alpha_m}} d\lambda
\end{aligned} \quad (3.8')$$

here $\alpha_m = \frac{\alpha}{m+1} < 1$. When estimating K' , we shall replace (3.11) by

$$(\lambda + (-\Delta)^{m+1})^{-1}(x, y) = \int_0^\infty e^{-t} t^{-\frac{n}{2m+2}} F\left(\frac{|x-y|}{t^{1/2(m+1)}}\right) dt,$$

where $F(\cdot) \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. It's then easy to see that estimate (3.9) is valid with $\frac{1}{p} - \frac{1}{q} = \frac{2\alpha}{n}$. In order to estimate K'' , we shall replace (3.13) by the following estimate

$$\begin{aligned}
& |(\lambda + (-\Delta)^{m+1})^{-1}(x, y)| \leq \\
& \begin{cases} C\lambda^{\sigma_1(m,n)} |x-y|^{\sigma_2(m,n)} \exp\{-\lambda^{1/2(2m+1)} |x-y|^{(m+1)/(2m+1)}\}, & \lambda^{1/2m} |x-y| > 1, \\ C|x-y|^{2m-n}, & \lambda^{1/2m} |x-y| \leq 1, \end{cases}
\end{aligned}$$

where $\sigma_1(m, n) = \frac{(4m+1)n+2(m+1)}{4(m+1)(2m+1)} - 1$, $\sigma_2(m, n) = \frac{2m+2-n}{2(2m+1)}$. This in turn follows from the fact that the heat kernel of the $e^{-t(-\Delta)^{m+1}}$ satisfies

$$|e^{-t(-\Delta)^{m+1}}(x, y)| \leq C t^{\frac{n}{2(m+1)}} \exp\left\{-C \frac{|x-y|^{\frac{2(m+1)}{2m+1}}}{t^{\frac{1}{2m+1}}}\right\}.$$

Now the desired estimates for K'' follows from the integer case ($\alpha = m+1$) and the above pointwise estimates of the resolvent of $(-\Delta)^{m+1}$. \square

Remark 3.1. We note that recently Cuenin [5] established a type of $L^p - L^{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$) uniform resolvent estimates for fractional Laplacian in order to study the eigenvalue

bounds for $H = (-\Delta)^\alpha + V$. However, our approach is complete different from the one in [5].

We point it out here that it follows from the proof of Theorem 1.4 that we also have the following $L^p - L^{p'}$ estimates, which generalize (1.1) to the fractional case.

Corollary 3.2. *Let $n \geq 3$, and $\alpha \geq \frac{n}{n+1}$, suppose $\frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{p'} \leq \frac{2\alpha}{n}$, then there is a uniform constant $C > 0$, such that*

$$\|((- \Delta)^\alpha - z)^{-1}\|_{L^p - L^{p'}} \leq C|z|^{\frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{p'}) - 1}, \quad z \in \mathbb{C} \setminus \{0\}. \quad (3.15)$$

3.2. Applications to perturbed fractional Schrödinger operator. First, we give a simple application to obtain uniform resolvent estimates when the L^p norm of the potential is small.

Proposition 3.3. *Assume $\frac{2n}{n+1} \leq 2\alpha < n$, $V \in L^{\frac{n}{2\alpha}}(\mathbb{R}^n)$, and let $H = (-\Delta)^\alpha + V$. There exists a constant $c_0 > 0$ such that if $\|V\|_{L^{\frac{n}{2\alpha}}} \leq c_0$, then*

$$\|(H - z)^{-1}\|_{L^p - L^{p'}} \leq C|z|^{\frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{p'}) - 1}, \quad z \in \mathbb{C} \setminus \{0\}$$

for $\max(\frac{2\alpha}{n}, \frac{n+3}{2(n+1)}) < \frac{1}{p} \leq \frac{n+2\alpha}{2n}$.

Proof. Note that we can choose q such that $(\frac{1}{p}, \frac{1}{q})$ satisfies (1.9). So we apply Theorem 1.4 and Hölder's inequality to obtain

$$\|V((- \Delta)^\alpha - z)^{-1}\|_{L^p - L^p} \leq C\|V\|_{L^{\frac{n}{2\alpha}}},$$

then one can choose $c_0 = \frac{1}{2C}$ to deduce

$$\sup_{|z|>0} \|(I + V((- \Delta)^\alpha - z)^{-1})^{-1}\|_{L^{p'} - L^{p'}} \leq 2 \quad (3.16)$$

Now the Proposition follows by combining (3.16) and (3.15). \square

Proposition 3.4. *Let $0 < 2m < n$, $m \in \mathbb{Z}$. Assume $V \in L^{\frac{n}{2m}}$ such that $\max\{\frac{n+1}{2n}, \frac{2m}{n}\} < \frac{2m}{n} + \frac{1}{p_0} < 1$ for some $p_0 > \frac{2n}{n-1}$. Consider the map $A(\lambda) = (\lambda - (-\Delta)^m)^{-1}V$, then we have the following*

(i) $A(\lambda)$ is a compact operator on $L^{p_0}(\mathbb{R}^n)$ for any $\lambda \in \mathbb{C}$;

(ii) $A(\lambda)$ is continuous from $\mathbb{C}_+ \setminus \{0\}$ to the space of bounded operators on $L^{p_0}(\mathbb{R}^n)$;

(iii) There exists some $E \subset \mathbb{R}$ with Lebesgue measure zero such that for any compact subinterval $K \subset \mathbb{R} \setminus E \setminus \{0\}$, we have

$$\sup_{\lambda \in K, 0 < \epsilon < 1} \|((- \Delta)^m + V - \lambda - i\epsilon)^{-1}\|_{L^{p_0} - L^{p_0}} < \infty, \quad (3.17)$$

where $\frac{n+3}{2(n+1)} < \frac{1}{p_0} \leq \frac{n+2m}{2n}$. In particular, $\sigma_{\text{sing}}((-\Delta)^m + V) \subset E$.

Proof. Note that we can choose $q > 1$ such that $(\frac{2m}{n} + \frac{1}{p_0}, \frac{1}{q})$ satisfies (1.9), hence Theorem 1.4 and our assumption on V imply that $A(\lambda)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for any $\lambda \in \mathbb{C}$. Without loss of generality, we can assume that $V \in L^\infty$ with compact support. Now suppose that $\{f_n\} \rightharpoonup 0$ in L^{p_0} . First, we observe that

$$(1 + (-\Delta)^m)A(\lambda) = V + (1 - \lambda)A(\lambda),$$

which yields that $A(\lambda)$ is bounded from L^{p_0} to the Sobolev space H^{2m, p_0} , then we can apply Rellich's compactness theorem to choose a subsequence f_{n_k} such that

$$\chi_{[|x| \leq R]} A(\lambda) f_{n_k} \rightarrow 0, \text{ in } L^{p_0}, \quad (3.18)$$

where χ is the characteristic function. On the other hand, we note that for fixed $\lambda \in \mathbb{C}$, when $|x - y|$ is large enough, we have

$$|(\lambda - (-\Delta)^m)^{-1}(x, y)| \leq \begin{cases} C|x - y|^{2m-n}, & \lambda = 0, \\ C|x - y|^{-\frac{n-1}{2}}, & \lambda \neq 0. \end{cases}$$

This in turn implies that

$$\|\chi_{[|x| \geq R]} A(\lambda) f\|_{L^{p_0}} \leq \begin{cases} CR^{2m-n+\frac{n}{p_0}} \|V\|_{L^\infty} \|f\|_{L^{p_0}}, & \lambda = 0, \\ CR^{-\frac{n-1}{2}+\frac{n}{p_0}} \|V\|_{L^\infty} \|f\|_{L^{p_0}}, & \lambda \neq 0. \end{cases}$$

Then for any $\epsilon > 0$, our assumption indicates that $p_0 > \max\{\frac{2n}{n-1}, \frac{n}{n-2m}\}$, and therefore we are allowed to choose a large enough constant R such that for all $\|f\|_{L^{p_0}} \leq 1$

$$\chi_{[|x| \geq R]} A(\lambda) f < \epsilon. \quad (3.19)$$

Combine (3.18) and (3.19), we prove (i).

To prove (ii), we may also assume $V \in L^\infty$, and $\text{supp } V \subset \{x \in \mathbb{R}^n, |x| \leq R\}$. Furthermore we note that by (3.2), it suffices to prove the case $m = 1$, which is contained in the proof of Theorem 1.4.

Having established (i) and (ii), we apply Fredholm theorem to see that

$$(I + (\lambda - (-\Delta)^m)^{-1} V)^{-1} : L^{p'} \rightarrow L^{p'}$$

exists on $\mathbb{R} \setminus E \setminus \{0\}$, where $m(E) = 0$. Moreover, $(I + (\lambda - (-\Delta)^m)^{-1} V)^{-1}$ is a continuous function of $\lambda \in \mathbb{C}_+ \setminus E \setminus \{0\}$. Now the desired estimates (3.17) follows from resolvent identity and (3.15). \square

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